MAT 534 FALL 2015 REVIEW FOR THE FINAL EXAM

General

The exam will be in class on Friday, December 11, 11:15am-1:45pm. It will consist of 8-9 problems and will be a closed book exam. In addition to the material covered in reviews for Midterms I and II, it will cover the following material.

Material covered in class after November 12, 2015

1) Finite-dimensional vector space V over a field F, $\mathrm{Hom}_F(V,W)$, representation of $T \in \mathrm{Hom}_F(V,W)$ by a matrix, $\mathrm{End}_F(V)$.
2) Tensor product of vector spaces $V \otimes W$ (over a field F), dual vector space V^*, $\mathrm{Hom}_F(V,W) = V^* \otimes W$, $T^* \in \mathrm{Hom}_F(W^*, V^*)$, column rank=row rank theorem.
3) Determinants and their properties.
4) Eigenvalue and eigenvectors, eigenspaces. Equivalent criteria for $\lambda \in F$ to be an eigenvalue for $T \in \mathrm{End}_F(V)$. Linear independence of eigenvectors corresponding to different eigenvalues. Characteristic polynomial $c_T(x)$. Diagonalizable T.
5) A finite-dimensional F-vector space with $T \in \mathrm{End}_F(V)$ as a pure torsion $F[x]$-module. Minimal polynomial $m_T(x)$ of T.
6) Invariant factor decomposition of a torsion $F[x]$-module V:
$$V \cong F[x]/(a_1) \oplus \cdots \oplus F[x]/(a_m),$$
where $a_1(x)|a_2(x)|\ldots|a_m(x)$ are invariant factors and $a_m(x) = m_T(x)$.
7) Companion matrix $C_{a(x)}$ for multiplication by $\bar{x} = x \bmod a(x)$ in the monomial basis of $F[x]/(a(x))$. Existence and uniqueness of a rational canonical form of $T \in \mathrm{End}_F(V)$.
8) The following are equivalent: $S, T \in \mathrm{End}_F(V)$ are similar; corresponding $F[x]$-modules are isomorphic; S and T have the same rational canonical form.
9) Characteristic polynomial $c_T(x)$ is the product of invariant factors, $c_T(T) = 0$ (Caley-Hamilton), and $c_T(x)$ divides some power of $m_T(x)$.
10) The Smith normal form of the matrix $xI - A$ by row and column operations over $F[x]$, where A is the matrix of T in a basis $e_1, \ldots, e_n$ of the vector space V. The algorithm in the

textbook (keeping track of the row operations) to establish the $F[x]$-module isomorphism

$$V \cong F[x]f_1 \oplus \cdots \oplus F[x]f_m, \quad F[x]f_i \simeq F[x]/(a_i(x)),$$

for finding the cyclic vectors $f_1, \dots, f_m \in V$. Completing each f_i to a basis $f_i, Tf_i, \dots, T^{\deg a_i - 1} f_i$ of the i-th cyclic subspace $V_i = F[x]/(a_i(x))$. Expressing this basis of V in terms of the basis $e_1, \dots, e_n$ gives a matrix P such that $P^{-1}AP$ is in rational canonical form.

11) From rational canonical form to Jordan canonical form under the main assumption that all roots of $c_T(x)$ are in the field F. Use Chinese Remainder theorem to pass from invariant factor decomposition to the elementary divisor decomposition. Jordan block of the multiplication by $\bar{x} = x \bmod (x-\lambda)^k$ in the basis $(\bar{x}-\lambda)^{k-1}, (\bar{x}-\lambda)^{k-2}, \dots, \bar{x}-\lambda, 1$ of $F[x]/(x-\lambda)^k$. Obtaining Jordan canonical form.
12) Necessary and sufficient criterion for T to be diagonalizable over F: all roots of $c_T(x)$ are in F and $m_T(x)$ has only simple roots.
13) Passing from cyclic vectors $f_1, \dots, f_m \in V$ to the basis of the Jordan canonical form: for each invariant factor

$$a(x) = a_j(x) = (x-\lambda_1)^{\alpha_1} \cdots (x-\lambda_k)^{\alpha_k}$$

and the cyclic vector $f = f_j$ consider the subspace V_{ij} of the cyclic space V_j consisting of vectors $v \in V_j$ such $(T - \lambda_i I)^{\alpha_i} v = 0$. It has the basis $(T-\lambda_i I)^{\alpha_i - 1} g_i, \dots, (T-\lambda_i I) g_i, g_i$, where

$$g_i = \frac{a(T)}{(T-\lambda_i)^{\alpha_i}} f, \quad i = 1, \dots, k.$$

Doing this for $j = 1, \dots, m$, and expressing the resulting basis of V in terms of the basis $e_1, \dots, e_n$ of V gives matrix P such that $P^{-1}AP$ is in the Jordan canonical form (here A is the matrix of T in the basis $e_1, \dots, e_n$).